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THERMOELASTIC CONTACT OF TWO CYLINDERS WITH NONSTATIONARY FRICTIONAL HEAT FORMATION

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The thermoelastic problem of the frictional interaction of two circular infinite cylinders with allowance for heat formation due to the action of time-dependent frictional forces changing along the general axis of the tribosystem is studied in an axisymmetric formulation.

1. Formulation of the Problem. We consider a tribosystem consisting of two circular hollow infinite cylinders inserted without clearance into one another (the axial section of the tribosystem is shown in Fig. 1). A circular cylinder with inside radius a_1 and outside radius a_0 is inserted into another cylinder of the same shape having inside radius a_0 and outside radius a_2. Radial stresses q_1 and q_2 depending on the axial coordinate and time are specified on the lateral surfaces of the two-layer packet.

We assume that one of the cylinders rotates about the other with time-varying angular velocity ω. Heat generation is caused by the action of frictional forces which occur at the contact surfaces of the cylinders and obey the Amonton law. The heat contact of the elements of the tribosystem is not ideal. Heat exchange according to the Newton law proceeds between the non-contacting surfaces of the packet and the ambient medium. Dynamic effects that can occur under the action of external load are ignored. We determine the temperature fields, heat fluxes, displacements, and stresses in the two-layer cylinder.

We relate this tribosystem to cylindrical coordinates by choosing a certain section as a zero section and directing the z axis along the cylinder axis. We assume that the behavior of the external load at infinity is such that one can use the Fourier integral transform with respect to the z coordinate. Since the external load does not depend on the angular coordinate θ, we study this problem in an axisymmetric formulation to determine the temperature fields, heat fluxes, thermoelastic stresses, and displacements.

With the above assumptions, the problem reduces to construction of solutions of the system containing the differential heat-conduction equations

$$\partial_r^2 T_j + r^{-1}\partial_r T_j + \partial_z^2 T_j = k_j^{-1}\partial_\tau T_j; \tag{1.1}$$

the equilibrium equations

$$\partial_r \sigma_r^{(j)} + r^{-1}(\sigma_r^{(j)} - \sigma_\theta^{(j)}) + \partial_z \tau_{rz}^{(j)} = 0, \quad \partial_r \tau_{rz}^{(j)} + r^{-1}\tau_{rz}^{(j)} + \partial_z \sigma_z^{(j)} = 0; \tag{1.2}$$

the compatibility equations

$$\partial_r \varepsilon_\theta^{(j)} + r^{-1}(\varepsilon_\theta^{(j)} - \varepsilon_r^{(j)}) = 0, \quad r\partial_z^2 \varepsilon_\theta^{(j)} + \partial_r \varepsilon_z^{(j)} = \partial_z \gamma_{rz}^{(j)}; \tag{1.3}$$

and the relations of Hooke's law

$$\begin{gathered} E_j\varepsilon_r^{(j)} = \sigma_r^{(j)} - \nu_j(\sigma_\theta^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j, \quad E_j\varepsilon_\theta^{(j)} = \sigma_\theta^{(j)} - \nu_j(\sigma_r^{(j)} + \sigma_z^{(j)}) + E_j\alpha_j T_j, \\ E_j\varepsilon_z^{(j)} = \sigma_z^{(j)} - \nu_j(\sigma_r^{(j)} + \sigma_\theta^{(j)}) + E_j\alpha_j T_j, \quad E_j\gamma_{rz}^{(j)} = 2(1+\nu_j)\tau_{rz}^{(j)} \quad (j = 1, 2), \end{gathered} \tag{1.4}$$

which are subject to the initial conditions

$$T_j(r, z, 0) = 0, \tag{1.5}$$

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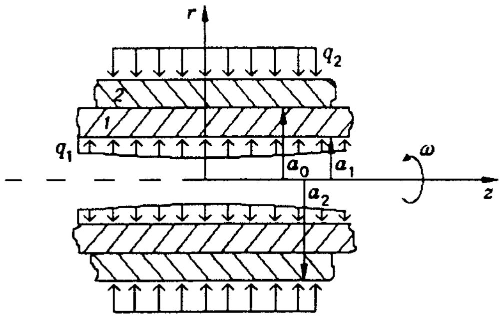

Fig. 1

and the boundary and contact conditions

$$r = a_1: \quad \partial_r T_1 = \gamma_1 T_1, \quad \sigma_r^{(1)} = -q_1(z,\tau), \quad \tau_{rz}^{(1)} = 0; \tag{1.6}$$

$$r = a_2: \quad \partial_r T_2 = -\gamma_2 T_2, \quad \sigma_r^{(2)} = -q_2(z,\tau), \quad \tau_{rz}^{(2)} = 0; \tag{1.7}$$

$$r = a_0: \quad \lambda_1 \partial_r T_1 - \lambda_2 \partial_r T_2 = f\omega(\tau) a_0 p(z,\tau), \quad \lambda_1 \partial_r T_1 + \lambda_2 \partial_r T_2 + h(T_1 - T_2) = 0,$$
$$\sigma_r^{(1)} = \sigma_r^{(2)} = -p(z,\tau), \qquad \tau_{rz}^{(1)} = \tau_{rz}^{(2)} = 0, \qquad u_r^{(1)} = u_r^{(2)}. \tag{1.8}$$

Here r and z are radial and axial coordinates, τ is time; $p(z,\tau)$ is the contact pressure; $\omega(\tau)$ is the relative angular velocity; T_j is the temperature; $\sigma_r^{(j)}$, $\sigma_\theta^{(j)}$, and $\sigma_z^{(j)}$ are the radial, tangential, and axial normal stresses; $\tau_{rz}^{(j)}$ is the tangential stress; $\varepsilon_r^{(j)}$, $\varepsilon_\theta^{(j)}$, and $\varepsilon_z^{(j)}$ are the radial, tangential, and axial linear strains; $\gamma_{rz}^{(j)}$ is the shear strain; $u_r^{(j)}$ is the radial displacement; E_j is the Young's modulus; ν_j, λ_j, k_j, and α_j are the Poisson's ratio, the heat conductivity, the thermal diffusivity, and the linear heat-expansion coefficient, respectively; $\gamma_j = \bar{\alpha}_j/\lambda_j$; $\bar{\alpha}_j$ is the heat-exchange coefficient; h is the heat conductivity of the contact surface; and f is the coefficient of friction. Here and below, $j = 1$ corresponds to the inner cylinder, and $j = 2$ to the external cylinder.

2. Construction of Solution. From the formulation of the problem it follows that the behavior of the solution depends on the distribution of the external load. We assume that the external load is distributed symmetrically about the section $z = 0$. Then, the solution of the formulated problem will also be symmetrical about the zero section. Then, we obtain additional symmetry conditions $[\partial_z T_j(r,0,\tau) = 0]$ and can use the Fourier integral cosine transform to solve the problem.

We reduce problem (1.1)–(1.8) to a system of two integral equations for the functions

$$f_j(z,\tau) = (-1)^{j-1} \partial_r T_j(a_0, z, \tau), \tag{2.1}$$

which are proportional to the heat fluxes on the contact surface. To this end, we express the temperature of the cylinders in terms of the functions f_j by first solving Eq. (1.1) subject to the initial conditions (1.5) and the thermal boundary conditions (1.6), (1.7), and (2.1). Taking the Fourier integral cosine transform [1] with respect to the axial z coordinate,

$$\bar{T}_j(r,\xi,\tau) = \int_0^\infty T_j(r,z,\tau)\cos(\xi z)\,dz$$

and using the Duhamel theorem [1] with respect to time τ, for the transformant of the temperature $\bar{T}_j$ we obtain the integral representation

$$\bar{T}_j(r,\xi,\tau) = \partial_\tau \int_0^\tau \bar{f}_j(\xi,y)\bar{\Phi}_j(r,\xi,\tau-y)\,dy, \tag{2.2}$$

where $\bar{f}_j(\xi,\tau)$ is the Fourier transformant of the function $f_j(z,\tau)$. The kernel of the integral representation is found by solution of the additional problem

$$\partial_r^2\bar{\Phi}_j + r^{-1}\partial_r\bar{\Phi}_j - \xi^2\bar{\Phi}_j = k_j^{-1}\partial_\tau\bar{\Phi}_j, \qquad \bar{\Phi}_j(r,\xi,0) = 0,$$
$$\partial_r\bar{\Phi}_j(a_0,\xi,\tau) = \pm 1, \qquad \partial_r\bar{\Phi}_j(a_j,\xi,\tau) = \pm\gamma_j\bar{\Phi}_j(a_j,\xi,\tau). \tag{2.3}$$

Here and below, in $\pm$ and $\mp$ the upper sign corresponds to $j = 1$ and the lower sign to $j = 2$.

The desired function $\bar{\Phi}_j(r,\xi,\tau)$ is written as the sum of two functions:

$$\bar{\Phi}_j(r,\xi,\tau) = \bar{\Phi}_{j1}(r,\xi) + \bar{\Phi}_{j2}(r,\xi,\tau). \tag{2.4}$$

The first term is a solution of the stationary heat-conduction equation and satisfies the inhomogeneous boundary conditions of problem (2.3):

$$\partial_r^2\bar{\Phi}_{j1} + r^{-1}\partial_r\bar{\Phi}_{j1} - \xi^2\bar{\Phi}_{j1} = 0, \quad \partial_r\bar{\Phi}_{j1}(a_0,\xi) = \pm 1, \quad \partial_r\bar{\Phi}_{j1}(a_j,\xi) = \pm\gamma_j\bar{\Phi}_{j1}(a_j,\xi). \tag{2.5}$$

The second term is a solution of the following boundary-value problem:

$$\partial_r^2\bar{\Phi}_{j2} + r^{-1}\partial_r\bar{\Phi}_{j2} - \xi^2\bar{\Phi}_{j2} = k_j^{-1}\partial_\tau\bar{\Phi}_{j2}, \qquad \bar{\Phi}_{j2}(r,\xi,0) = -\bar{\Phi}_{j1}(r,\xi),$$
$$\partial_r\bar{\Phi}_{j2}(a_0,\xi,\tau) = 0, \quad \partial_r\bar{\Phi}_{j2}(a_j,\xi,\tau) = \pm\gamma_j\bar{\Phi}_{j2}(a_j,\xi,\tau). \tag{2.6}$$

Thus, we obtained a Bessel differential equation for the functions $\bar{\Phi}_{j1}(r,\xi)$. Satisfying the boundary conditions, we have

$$\bar{\Phi}_{j1}(r,\xi) = \pm\xi^{-1}\frac{I_0(\xi r)[\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)] + K_0(\xi r)[\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j)]}{I_1(\xi a_0)[\xi K_1(\xi a_j) \pm \gamma_j K_0(\xi a_j)] - K_1(\xi a_0)[\xi I_1(\xi a_j) \mp \gamma_j I_0(\xi a_j)]}, \tag{2.7}$$

where $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions of order ν of the first and second kinds [2].

To solve problem (2.6), we take the Hankel finite integral transform [1] with respect to the radial coordinate r:

$$\tilde{\Phi}_{j2}(\mu_{j,m},\xi,\tau) = \pm\int_{a_j}^{a_0} r\bar{\Phi}_{j2}(r,\xi,\tau)K_j(r,\mu_{j,m})\,dr.$$

Here

$$K_j(r,\mu_{j,m}) = W_0(\mu_{j,m}r,\mu_{j,m}a_0)N_{j,m}^{-1} \qquad (m = 1,2,\ldots) \tag{2.8}$$

is the orthonormalized kernel determined from the solution of the Sturm–Liouville problem: $\partial_r^2 K_j + r^{-1}\partial_r K_j + \mu_j^2 K_j = 0$, where $\partial_r K_j = 0$ for $r = a_0$ and $\partial_r K_j = \pm\gamma_j K_j$ for $r = a_j$, $\mu_{j,m}$ are roots of the characteristic equations $\mu_j W_1(\mu_j a_j,\mu_j a_0) \pm \gamma_j W_0(\mu_j a_j,\mu_j a_0) = 0$; $N_{j,m}$ is a normalizing factor; and $N_{j,m}^2 = \pm 0.5[a_0^2 W_0^2(\mu_{j,m}a_0,\mu_{j,m}a_0) - a_j^2[W_0^2(\mu_{j,m}a_j,\mu_{j,m}a_0) + W_1^2(\mu_{j,m}a_j,\mu_{j,m}a_0)]]$. In the previous relations, we introduced the functions

$$W_0(x,y) = J_0(x)Y_1(y) - Y_0(x)J_1(y), \quad W_1(x,y) = J_1(x)Y_1(y) - Y_1(x)J_1(y),$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of order ν of the first and second kinds [2].

Applying the integral transform to problem (2.6) and solving the resulting first-order ordinary differential equation, we obtain a formula for determining the Hankel transformant of the function $\bar{\Phi}_{j2}(r,\xi,\tau)$:

$$\tilde{\Phi}_{j2}(\mu_{j,m},\xi,\tau) = -\tilde{\Phi}_{j1}(\mu_{j,m},\xi)\exp[-k_j(\xi^2 + \mu_{j,m}^2)\tau]. \tag{2.9}$$

Here

$$\tilde{\Phi}_{j1}(\mu_{j,m},\xi) = a_0 W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)((\xi^2 + \mu_{j,m}^2)N_{j,m})^{-1} \tag{2.10}$$

is the Hankel transformant of the function $\bar{\Phi}_{j1}(r,\xi)$. Inversion of the Hankel transformant is performed by the formula

$$\bar{\Phi}_{j2}(r,\xi,\tau) = \sum_{m=1}^{\infty}\tilde{\Phi}_{j2}(\mu_{j,m},\xi,\tau)K_j(r,\mu_{j,m}).$$

Using relations (2.8)-(2.10), we write the solution of problem (2.6) in the form

$$\bar{\Phi}_{j2}(r,\xi,\tau) = -a_0 \sum_{m=1}^{\infty} \frac{W_0(\mu_{j,m}r, \mu_{j,m}a_0)W_0(\mu_{j,m}a_0, \mu_{j,m}a_0)}{N_{j,m}^2(\xi^2+\mu_{j,m}^2)} \exp[-k_j(\xi^2+\mu_{j,m}^2)\tau]. \tag{2.11}$$

Thus, formulas (2.2), (2.4), (2.7), and (2.11) give the Fourier transformants of the temperature of each of the cylinders. For $\xi = 0$, these formulas remain valid, if one takes into account that $\bar{\Phi}_{j1}(r,0) = \pm a_0 \ln(r/a_j) + a_0[a_j\gamma_j]^{-1}$.

The thermoelastic problem (1.2)-(1.4) under stresses specified on the boundary [boundary conditions (1.6)-(1.8)] is solved by the method described in [3, 4]. For each cylinder we introduce the stress function $\varphi_j(r,z,\tau)$, and the problem reduces to the inhomogeneous biharmonic equation

$$DD\varphi_j + 2D\partial_z^2\varphi_j + \partial_z^4\varphi_j = F_j(r,z,\tau) \tag{2.12}$$

subject to the homogeneous boundary conditions

$$D\varphi_j = 0, \qquad \partial_r\varphi_j = r^{-1}(1-\nu_j)\varphi_j, \qquad r = a_0, a_j. \tag{2.13}$$

In relations (2.12) and (2.13), D is the differential operator $r\partial_r r^{-1}\partial_r$:

$$F_j(r,z,\tau) = -\frac{a_j^2}{a_0^2-a_j^2}\Big(\frac{a_0^2}{1-\nu_j}+\frac{r^2}{1+\nu_j}\Big)\Big[\partial_z^2 q_j(z,\tau) + \frac{E_j\alpha_j}{a_j}\partial_r T_j(a_j,z,\tau) + \frac{E_j\alpha_j}{a_j^2}k_j^{-1}\partial_\tau \int_{a_j}^{a_0} \rho T_j(\rho,z,\tau)\,d\rho\Big]$$

$$+\frac{a_0^2}{a_0^2-a_j^2}\Big(\frac{a_j^2}{1-\nu_j}+\frac{r^2}{1+\nu_j}\Big)\Big[\partial_z^2 p(z,\tau) + \frac{E_j\alpha_j}{a_0}\partial_r T_j(a_0,z,\tau)\Big] - \frac{E_j\alpha_j}{1-\nu_j}k_j^{-1}\partial_\tau \int_{a_0}^{r} \rho T_j(\rho,z,\tau)\,d\rho.$$

Radial displacements are given by the formula

$$u_r^{(j)}(r,z,\tau) = -\frac{1-\nu_j^2}{E_j}r^{-1}\Big[-\frac{q_j(z,\tau)a_j^2}{a_0^2-a_j^2}\Big(\frac{a_0^2}{1-\nu_j}+\frac{r^2}{1+\nu_j}\Big)+\frac{p(z,\tau)a_0^2}{a_0^2-a_j^2}\Big(\frac{a_j^2}{1-\nu_j}+\frac{r^2}{1+\nu_j}\Big)-\frac{E_j\alpha_j}{a_0^2-a_j^2}\Big(\frac{a_0^2}{1-\nu_j}+\frac{r^2}{1+\nu_j}\Big)$$

$$\times\int_{a_j}^{a_0}\rho T_j(\rho,z,\tau)\,d\rho - \frac{E_j\alpha_j}{1-\nu_j}\int_{a_0}^{r}\rho T_j(\rho,z,\tau)\,d\rho - \partial_z^2\varphi_j(r,z,\tau)\Big] - \frac{\nu_j(1+\nu_j)}{E_j}r^{-1}D\varphi_j(r,z,\tau). \tag{2.14}$$

Applying the Fourier integral cosine transform to Eq. (2.12) and boundary conditions (2.13), for the transformant $\bar{\varphi}_j(r,\xi,\tau)$ we obtain a fourth-order inhomogeneous differential equation. The solution of this equation is found as the sum of the general solution of the homogeneous equation and the partial solution of the inhomogeneous equation. The general solution of the homogeneous equation is represented by a linear combination of zero- and first-order modified Bessel functions of the first and second kind. To determine the four unknown coefficients of the general solution, we use the boundary conditions. Satisfaction of these conditions leads to a system of four linear algebraic equations. Solving the system, we find a formula for the transformant of the stress function $\bar{\varphi}_j(r,\xi,\tau)$ for each cylinder. Having the formula of the transformant of the stress function, we can write formulas for the transformant of the displacements, strains, and stresses.

Here we write only the expression for the transformant of radial displacements on the contact surface of the cylinders:

$$\bar{u}_r^{(j)}(a_0,\xi,\tau) = \frac{1-\nu_j^2}{E_j}\Big(a_0\bar{p}(\xi,\tau)\frac{\Delta_1(a_j,\xi)}{\Delta(a_j,\xi)} - a_j\bar{q}_j(\xi,\tau)\frac{\Delta_2(a_j,\xi)}{\Delta(a_j,\xi)}\Big) + \partial_\tau\int_0^{\tau}\bar{f}_j(\xi,y)\bar{H}_j(\xi,\tau-y)\,dy, \tag{2.15}$$

where

$$\bar{H}_j(\xi,\tau) = \bar{H}_{j1}(\xi) + \bar{H}_{j2}(\xi,\tau);$$

$$\bar{H}_{j1}(\xi) = \frac{\alpha_j(1-\nu_j^2)}{\xi^2}\Big[\frac{\Delta_2(a_j,\xi)}{\Delta(a_j,\xi)}\partial_r\bar{\Phi}_{j1}(a_j,\xi) \mp \Big(\frac{\Delta_1(a_j,\xi)}{\Delta(a_j,\xi)} - \frac{1}{1-\nu_j}\Big)\Big];$$

$$\bar{H}_{j2}(\xi,\tau) = \alpha_j(1+\nu_j)a_0 \sum_{m=1}^{\infty} \frac{W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)}{N_{j,m}^2(\xi^2+\mu_{j,m}^2)^2}\Big[\xi^2\Big(\frac{\Delta_1(a_j,\xi)}{\Delta(a_j,\xi)}a_0 W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)$$

$$-\frac{\Delta_2(a_j,\xi)}{\Delta(a_j,\xi)}a_j W_0(\mu_{j,m}a_j,\mu_{j,m}a_0)\Big)+\frac{\Delta_3(a_j,\xi)}{\Delta(a_j,\xi)}\mu_{j,m}W_1(\mu_{j,m}a_j,\mu_{j,m}a_0)\Big]\exp[-k_j(\xi^2+\mu_{j,m}^2)\tau];$$

$$\Delta(a_j,\xi) = 4(1-\nu_j)+a_j^2\xi^2+a_0^2\xi^2+(2(1-\nu_j)+a_j^2\xi^2)(2(1-\nu_j)+a_0^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi)$$

$$-I_1(a_0\xi)K_1(a_j\xi)]^2 - a_j^2\xi^2(2(1-\nu_j)+a_0^2\xi^2)[I_0(a_j\xi)K_1(a_0\xi)+I_1(a_0\xi)K_0(a_j\xi)]^2$$

$$-a_0^2\xi^2(2(1-\nu_j)+a_j^2\xi^2)[I_1(a_j\xi)K_0(a_0\xi)+I_0(a_0\xi)K_1(a_j\xi)]^2 + a_j^2a_0^2\xi^4[I_0(a_j\xi)K_0(a_0\xi)-I_0(a_0\xi)K_0(a_j\xi)]^2;$$

$$\Delta_1(a_j,\xi) = 2[1+(2(1-\nu_j)+a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi)-I_1(a_0\xi)K_1(a_j\xi)]^2$$

$$-a_j^2\xi^2[I_0(a_j\xi)K_1(a_0\xi)+I_1(a_0\xi)K_0(a_j\xi)]^2];$$

$$\Delta_2(a_j,\xi) = 2a_0\xi[I_1(a_j\xi)K_0(a_0\xi)+I_0(a_0\xi)K_1(a_j\xi)] - 2a_j\xi[I_0(a_j\xi)K_1(a_0\xi)+I_1(a_0\xi)K_0(a_j\xi)];$$

$$\Delta_3(a_j,\xi) = 2[(2(1-\nu_j)+a_j^2\xi^2)[I_1(a_j\xi)K_1(a_0\xi)-I_1(a_0\xi)K_1(a_j\xi)] - a_ja_0\xi^2[I_0(a_j\xi)K_0(a_0\xi)-I_0(a_0\xi)K_0(a_j\xi)]].$$

Using the condition of equality of radial displacements on the contact surface and relations (2.15), we obtain integral representations for the transformant of the contact pressure:

$$\bar{p}(\xi,\tau) = \Big[\sum_{k=1}^{2}(-1)^{k-1}\Big(\frac{1-\nu_k^2}{E_k}a_k\bar{q}_k(\xi,\tau)\frac{\Delta_2(a_k,\xi)}{\Delta(a_k,\xi)} - \partial_\tau\int_0^\tau \bar{f}_k(\xi,y)\bar{H}_k(\xi,\tau-y)dy\Big)\Big]$$

$$\times\Big[a_0\sum_{k=1}^{2}(-1)^{k-1}\frac{1-\nu_k^2}{E_k}\frac{\Delta_1(a_k,\xi)}{\Delta(a_k,\xi)}\Big]^{-1}. \tag{2.16}$$

In this case,

$$\bar{p}(0,\tau) = \Big[\sum_{k=1}^{2}(-1)^{k-1}\Big(\frac{\bar{q}_k(0,\tau)}{E_k}\frac{2a_k^2}{a_0^2-a_k^2} - \partial_\tau\int_0^\tau \bar{f}_k(0,y)\bar{H}_k(0,\tau-y)\,dy\Big)\Big]$$

$$\times\Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{1}{E_k}\Big(\frac{a_0^2+a_k^2}{a_0^2-a_k^2}-\nu_k\Big)\Big]^{-1}.$$

Here

$$\bar{H}_{j1}(0) = \alpha_j a_0\Big(\mp\frac{a_0^2}{a_0^2-a_j^2}\ln\Big(\frac{a_0}{a_j}\Big) \pm \Big(0.5\mp\frac{1}{a_j\gamma_j}\Big)\Big);$$

$$\bar{H}_{j2}(0,\tau) = -\frac{2\alpha_ja_ja_0}{a_0^2-a_j^2}\sum_{m=1}^{\infty}\frac{W_1(\mu_{j,m}a_j,\mu_{j,m}a_0)W_0(\mu_{j,m}a_0,\mu_{j,m}a_0)}{N_{j,m}^2\mu_{j,m}^3}\exp[-k_j\mu_{j,m}^2\tau].$$

Applying the Fourier integral cosine transform to the thermal contact conditions (1.8) and using relations (2.1), (2.2), and (2.16), we obtain a system of Volterra integral equations of the second kind for the Fourier transformants of the functions $f_j(z,\tau)$:

$$\sum_{k=1}^{2}\lambda_k\bar{f}_k(\xi,\tau) = f\omega(\tau)\Big[\sum_{k=1}^{2}(-1)^{k-1}\Big(\frac{1-\nu_k^2}{E_k}a_k\bar{q}_k(\xi,\tau)\frac{\Delta_2(a_k,\xi)}{\Delta(a_k,\xi)}$$

$$-\partial_\tau\int_0^\tau \bar{f}_k(\xi,y)\bar{H}_k(\xi,\tau-y)\,dy\Big)\Big]\Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{1-\nu_k^2}{E_k}\frac{\Delta_1(a_k,\xi)}{\Delta(a_k,\xi)}\Big]^{-1}, \tag{2.17}$$

$$\sum_{k=1}^{2}(-1)^{k-1}\Big[\lambda_k\bar{f}_k(\xi,\tau)+h\partial_\tau\int_0^\tau \bar{f}_k(\xi,y)\bar{\Phi}_k(a_0,\xi,\tau-y)\,dy\Big] = 0.$$

Solving this system of equation, we find the transformants of the contact pressure and the functions $f_j(z,\tau)$. Together with the formulas for the transformants of the stress function $\bar{\varphi}_j(r,\xi,\tau)$ they give the complete analytical solution of the formulated problem in the transformants of the Fourier integral cosine transform.

The original is given by the inversion formula [1]

$$\varphi_j(r,z,\tau) = 2\pi^{-1}\int_0^\infty \bar{\varphi}_j(r,\xi,\tau)\cos(z\xi)\,d\xi, \tag{2.18}$$

and the value of the integral is found numerically.

3. Construction of Numerical Algorithm. System (2.17) is solved numerically. The behavior of the tribosystem is studied in the interval $[0,\tau^*]$, which is divided into N parts with constant step τ_1. Then, at each time $\tau_n = n\tau_1$ $(n = 1,\ldots,N)$, the integrals are replaced by finite sums according to the scheme

$$\int_0^{\tau_n} f(y)F(\tau_n - y)\,dy = \tau_1\Big(0.5f(0)F(\tau_n) + \sum_{i=1}^{n-1} f(\tau_i)F(\tau_n-\tau_i)\Big),$$

where

$$\int_0^{\tau_1} f(y)F(\tau_1 - y)\,dy = 0.5\tau_1 f(0)F(\tau_1).$$

These relations were written using the quadrature trapezoidal formula, and the mode of behavior of the functions $\bar{\Phi}_j(r,\xi,\tau)$ and $\bar{H}_j(\xi,\tau)$: by construction, [problem (2.3)] $\bar{\Phi}_j(r,\xi,0) = 0$, and the function $\bar{H}_j(\xi,\tau)$ has the same value for $\tau = 0$. The derivatives of the functions are replaced by finite differences according to the rule $d_\tau T(\tau_n) = 0.5\tau_1^{-1}(T(\tau_{n+1}) - T(\tau_{n-1}))$.

Then, the derivatives of integrals in system (2.17) are replaced by the relations

$$I_n = \Big(\partial_\tau \int_0^\tau f(y)F(\tau-y)\,dy\Big)_{\tau=\tau_n} = 0,25f(0)\,[F(\tau_{n+1}) - F(\tau_{n-1})]$$

$$+0.5\Big(f(\tau_n)F(\tau_1) + f(\tau_{n-1})F(\tau_2) + \sum_{i=1}^{n-2} f(\tau_i)[F(\tau_{n+1}-\tau_i) - F(\tau_{n-1}-\tau_i)]\Big) \quad (n = 3,4,\ldots),$$

$$I_1 = 0.25f(0)F(\tau_2) + 0.5f(\tau_1)F(\tau_1), \quad I_2 = 0.25f(0)[F(\tau_3) - F(\tau_1)] + 0.5(f(\tau_1)F(\tau_2) + f(\tau_2)F(\tau_1)).$$

It is easy to show that $I_0 = 0$.

Analysis of these relations shows that, at each time τ_n, the system of integral equations (2.17) reduces to a system of two linear algebraic equations for functions $\bar{f}_j(\xi,\tau_n)$. The functions $\bar{f}_j$ are linear functions of the external load: $\bar{f}_j(\xi,\tau_n) = \bar{Q}_{j1}(\xi,\tau_n)\bar{q}_1(\xi,\tau_n) + \bar{Q}_{j2}(\xi,\tau_n)\bar{q}_2(\xi,\tau_n)$. Similar relationships hold for the transformants of the temperature, contact pressure, and the stress function.

Numerical analysis shows that the functions $\bar{Q}_{jk}(\xi,\tau_n)$ are not oscillating and decrease fairly rapidly as ξ increases. Therefore, in the inversion of the functions $\bar{f}_j(\xi,\tau_n)$, we can divide the interval of integration $[0,\infty)$ into two parts, $[0,\xi^*]$ and $[\xi^*,\infty)$, where ξ^* is of the order of 1000, and ignore the integral over the second interval. To calculate the integral over the interval $[0,\xi^*]$, we use Filon's method [5] (the method of approximate calculation of integrals of trigonometric functions); the behavior of the functions $\bar{q}_j$ should be taken into account. We explain this by an example.

Let $q_1(z,\tau) = 0$ and $q_2(z,\tau) = q_0(\tau)H(L-z)H(L+z)$, where $H(z)$ is the Heaviside function [1], and L is a parameter that determines the interval of application of the external load. In this case, the Fourier transformant of the external load has the form $\bar{q}_2(\xi,\tau) = q_0(\tau)\sin(L\xi)\xi^{-1}$. After additional manipulations we obtain

$$f_j(z,\tau) = 2\pi^{-1}q_0(\tau)\int_0^{\xi^*} \bar{Q}_{j2}(\xi,\tau)\xi^{-1}\sin(L\xi)\cos(z\xi)\,d\xi$$

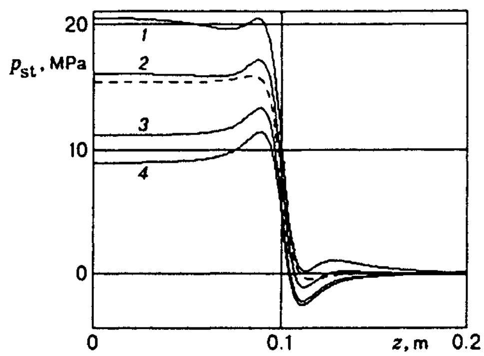

Fig. 2

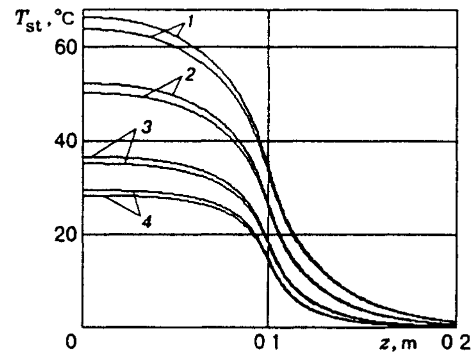

Fig. 3

$$= \pi^{-1} q_0(\tau) \int_0^{\xi^*} [\bar{Q}_{j2}(\xi,\tau) - \bar{Q}_{j2}(0,\tau)]\xi^{-1}[\sin(\xi(L+z)) + \sin(\xi(L-z))]\,d\xi$$
$$+\bar{Q}_{j2}(0,\tau)\pi^{-1}q_0(\tau)[\,\mathrm{Si}(\xi^*(L+z)) + \mathrm{Si}(\xi^*(L-z))],$$

where

$$\mathrm{Si}(z) = \int_0^z \sin(x)x^{-1}\,dx$$

is the integral sine [2], and we can use directly Filon's formulas.

Numerical studies show that for a relative calculation error of 1%, it suffices to divide the interval $[0,\xi^*]$ into segments with a step of 5 m^{-1} and take τ_1 equal to 2 sec.

4. Analysis of the Results. Analysis of the previous results shows that with time the contact pressure (2.16) reaches a stationary value that coincides with the analytical contact pressure of the stationary problem.

An expression for the transformant of the contact pressure of the stationary problem can be found from relation (2.16) and system (2.17) by applying to them the Laplace integral transform and using the relation

$$\lim_{\tau\to\infty} \psi(\tau) = \lim_{s\to 0} s\bar{\psi}(s),$$

where $\bar{\psi}(s)$ is the Laplace transform of the function $\psi(\tau)$ [1]. For example, provided that $q_j(z,\tau) = q_j^*(z)(1 - \exp(-\beta\tau))$ and $\omega = \omega^* = \mathrm{const}$, we obtain the formula

$$\bar{p}_{st}(\xi) = \Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{1-\nu_k^2}{E_k}\frac{a_k}{a_0}\bar{q}_k^*(\xi)\frac{\Delta_2(a_k,\xi)}{\Delta(a_k,\xi)}\Big]\Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{1-\nu_k^2}{E_k}\frac{\Delta_1(a_k,\xi)}{\Delta(a_k,\xi)} - f\omega^*\bar{R}(\xi)\Big]^{-1}, \qquad (4.1)$$

where

$$\bar{R}(\xi) = \frac{\bar{H}_{21}(\xi)(\lambda_1 + h\bar{\Phi}_{11}(a_0,\xi)) - \bar{H}_{11}(\xi)(\lambda_2 + h\bar{\Phi}_{21}(a_0,\xi))}{\lambda_2(\lambda_1 + h\bar{\Phi}_{11}(a_0,\xi)) + \lambda_1(\lambda_2 + h\bar{\Phi}_{21}(a_0,\xi))}.$$

For $\xi = 0$ we have

$$\bar{p}_{st}(0) = \Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{\bar{q}_k^*(0)}{E_k}\frac{2a_k^2}{a_0^2 - a_k^2}\Big]\Big[\sum_{k=1}^{2}(-1)^{k-1}\frac{1}{E_k}\Big(\frac{a_0^2 + a_k^2}{a_0^2 - a_k^2} - \nu_k\Big) - f\omega^* a_0\bar{R}(0)\Big]^{-1}.$$

The structure of the denominator of expression (4.1) shows that, for each value ξ from the interval $[0,\infty)$, there is a critical value of the angular velocity ω^* for which the transformant of the contact pressure $\bar{p}_{st}(\xi)$ becomes infinitely large. Therefore, one can calculate the contact pressure of the stationary problem only for values of ω^* smaller than $\min \omega_{cr}^*(\xi)$. For example, for $\alpha_1 = \alpha_2 = 12\cdot 10^{-6}$ K^{-1}, we have $\min\omega_{cr}^* \approx 3.32$ sec^{-1}. An increase in the parameter α_1 decreases $\min\omega_{cr}^*$, and a decrease in α_1 increases $\min\omega_{cr}^*$. The other parameters in the numerical calculations were as follows: $E_j = 2\cdot 10^5$ MPa, $\nu_j = 0.3$, $\lambda_j = 50$ W/(m·K),

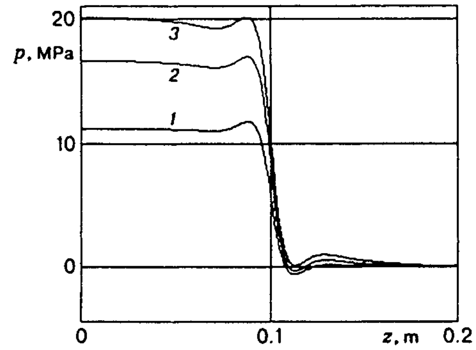

Fig. 4

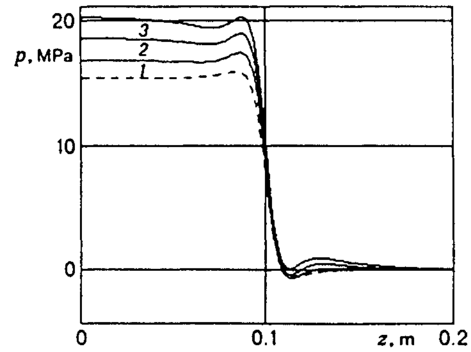

Fig. 5

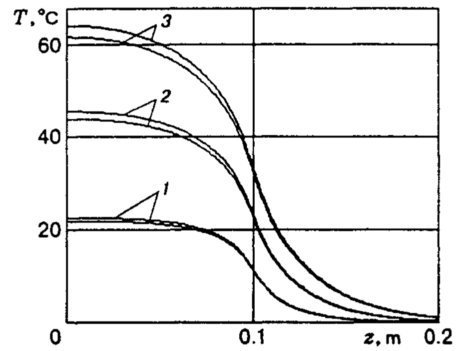

Fig. 6

$\bar{\alpha}_j = 1$ kW/(m$^2\cdot$ K), $k_j = 1.25 \cdot 10^{-5}$ m^2/sec, $h = 10$ kW/(m$^2\cdot$ K), $f = 0.1$, $a_1 = 3.5$ cm, $a_2 = 6$ cm, $a_0 = 5$ cm, $\alpha_2 = 12 \cdot 10^{-6}$ K^{-1}, $\alpha_1 = 1.2$, 6, 12, and $15 \cdot 10^{-6}$ K^{-1}, $\omega^* = 1$ and 2 sec^{-1}. The external load was varied as $q_1^*(z) = 0$, $q_2^*(z) = q^* H(L - z) H(L + z)$, where $q^* = 20$ MPa and $L = 0,1$ m.

The dependence of the contact pressure of the stationary problem on the linear thermal expansion coefficients is not chosen arbitrarily. The numerical study has shown that an increase in the parameter α_1 causes an increase in the contact pressure in the thermoelastic problem with respect to the contact pressure in the purely elastic problem. In this case, the increase in the heat-formation intensity due to increase in ω^* increases the contact pressure. A decrease in the linear thermal expansion coefficient α_1 causes opposite effects.

Figure 2 shows the variation in the contact pressure along the cylinder axis versus the parameter α_1 in the stationary problem. Curves 1–4 correspond to $\alpha_1 = 15$, 12, 6, and $1.2 \cdot 10^{-6}$ K^{-1} ($\alpha_2 = 12 \cdot 10^{-6}$ K^{-1} and $\omega^* = 1$ sec^{-1}), and the dashed curve shows the contact pressure of the elastic problem. The existence of a separation zone for this loading pattern is explained by the discontinuous character of the external load. Note that an increase in α_1 causes expansion of the inner cylinder, so that no separation is observed even under a discontinuous external load.

Figure 3 shows the distribution of the steady contact temperature along the axis of the two-layer cylinder. The curve numbers correspond to the values of α_1 in Fig. 2 (the upper curve refers to the first body, and the lower curve to the second body). The temperature jump on the surface $r = a_0$ is caused by imperfect thermal contact.

In the calculations of the contact pressure of the nonstationary problem, two patterns of variation of the external load and the angular velocity were chosen:

(1) $q_1(z, \tau) = 0$, $q_2(z, \tau) = q_2^*(z)(1 - \exp(-\beta\tau))$, and $\omega(\tau) = \omega^*$;

(2) $q_1(z, \tau) = 0$, $q_2(z, \tau) = q_2^*(z)$, and $\omega(\tau) = \omega^*(1 - \exp(-\beta\tau))$;

where $\beta = 0.01$ sec^{-1}. Numerical studies show that the time during which the tribosystem reaches a stationary value is about 900 sec in both cases.

Figures 4 and 5 show graphs of variation in the contact pressure p along the axis of the two-layer tribosystem for some times (Fig. 4 corresponds to the first pattern of variation in the external load and angular velocity, and Fig. 5 to the second). Curves 1–3 in Figs. 4 and 5 correspond to $\tau = 200$, 400, and 900 sec ($\alpha_1 = 15 \cdot 10^{-6}$ K^{-1}, $\alpha_2 = 12 \cdot 10^{-6}$ K^{-1}, and $\omega^* = 1$ sec^{-1}). The dashed curve in Fig. 5 refers to the contact pressure for $\tau = 0$. The contact pressure reaches monotonically a stationary value, provided that the angular velocity $\omega(\tau)$ varies in the interval $[0, \omega_0]$ ($\omega_0 < \min\ \omega^*_{cr}$).

Figure 6 shows the distribution of the contact temperature for the same times as in Figs. 4 and 5. The temperature of the cylinders for the above values of the external load and angular velocity reaches monotonically a stationary value. In this case, there is an insignificant difference in the character of the distribution and the temperature values obtained for the above dependences.

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